

On the Zeros of the Error Function for Tchebycheff Approximation on a Disk

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For the case of best Tchebycheff approximation to a real continuous function on an interval $[a, b]$ by a polynomial of degree n it is well-known that the error function has at least $n + 1$ zeros on $[a, b]$. No analogous result, however, is known for the case of Tchebycheff approximation to an analytic function on a disk of the complex plane. Indeed it is easy to construct a function f analytic on $\Delta: |z| \leq 1$ for which the constant of best uniform approximation to f on Δ is not in the range $f(\Delta)$. On the other hand, there are known examples [1, 4] of functions f for which the n -th error function has precisely $n + 1$ zeros in Δ .

The purpose of this paper is to exhibit a class of entire functions f , which includes the exponential function, with the property that for each n sufficiently large the polynomial of degree n of best Tchebycheff approximation to f on Δ interpolates to f in precisely $n + 1$ points in Δ (Theorem 5). We also study the distribution of these interpolation points (Theorem 4).

In [3] Motzkin and Walsh also investigate the zeros of the n -th error function, but their results pertain to Tchebycheff approximation on small disks whose radii depend on n .

Throughout this paper $f(z)$ denotes a function analytic on Δ , $p_n(z)$ is the polynomial of degree n of best Tchebycheff approximation to f on Δ , and

$$E_n(f) \equiv \|f - p_n\| \equiv \max_{|z| \leq 1} |f(z) - p_n(z)|.$$

We begin with a result on the degree of best approximation.

THEOREM 1. *Let $f(z) = \sum_0^\infty a_k z^k$, where $a_{n+1}/a_n \rightarrow 0$ as $n \rightarrow \infty$. Then*

$$E_n(f) = |a_{n+1}| [(1 + 4 |a_{n+2}/a_{n+1}|^2)^{1/2} + O(|a_{n+3}/a_{n+1}|)] \quad (1)$$

as $n \rightarrow \infty$.

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Proof. Let $\lambda = e^{i\phi}$, where $\phi = -\arg(a_{n+2}/a_{n+1})$, and set

$$f_1(z) \equiv f(\lambda z), \quad q(z) \equiv \sum_0^n a_k \lambda^k z^k + a_{n+2} \lambda^{n+2} z^n.$$

Then we have

$$\begin{aligned} E_n(f) &= E_n(f_1) \leq \|f_1 - q\| \\ &\leq |a_{n+1}| \left\| z + \frac{a_{n+2}}{a_{n+1}} \lambda z^2 - \frac{a_{n+2}}{a_{n+1}} \lambda \right\| + \sum_{n+3}^{\infty} |a_k|. \end{aligned} \quad (2)$$

It is trivial to verify that

$$\|z + \tau z^2 - \tau\| = (1 + 4\tau^2)^{1/2}, \quad \tau \text{ real,}$$

and hence from (2) we obtain

$$E_n(f) \leq |a_{n+1}| \left[(1 + 4|a_{n+2}/a_{n+1}|^2)^{1/2} + |a_{n+3}/a_{n+1}| \sum_{n+3}^{\infty} |a_k/a_{n+3}| \right].$$

This last inequality implies (1).

For comparison we mention the lower estimate

$$|a_{n+1}| (1 + |a_{n+2}/a_{n+1}|^2)^{1/2} \leq E_n(f),$$

which follows from the inequalities

$$\sum_{n+1}^{\infty} |a_k|^2 \leq \frac{1}{2\pi} \int_{|z|=1} |f(z) - p_n(z)|^2 dz \leq E_n(f)^2.$$

The above methods give the bounds (compare [2, p. 80])

$$\begin{aligned} \left(1 + \frac{1}{(n+2)^2}\right)^{1/2} &\leq (n+1)! E_n(e^z) \\ &\leq \left(1 + \frac{4}{(n+2)^2}\right)^{1/2} + \frac{n+4}{(n+2)(n+3)^2}, \\ \left(1 + \frac{1}{(2n+3)^2(2n+4)^2}\right)^{1/2} &\leq (2n+2)! E_{2n}(\cos z) \\ &\leq \left(1 + \frac{4}{(2n+3)^2(2n+4)^2}\right)^{1/2} \\ &\quad + \left(\frac{56}{55}\right) \frac{(2n+2)!}{(2n+6)!}, \\ \left(1 + \frac{1}{(2n+4)^2(2n+5)^2}\right)^{1/2} &\leq (2n+3)! E_{2n+1}(\sin z) \\ &\leq \left(1 + \frac{4}{(2n+4)^2(2n+5)^2}\right)^{1/2} \\ &\quad + \left(\frac{72}{71}\right) \frac{(2n+3)!}{(2n+7)!}. \end{aligned}$$

There are certain analytic functions f for which the polynomials $p_n(z)$ of best Tchebycheff approximation on Δ turn out to be the sections of the Taylor development for f (see [5]). This is not the case for the functions of Theorem 1. Indeed we have the following.

COROLLARY. *Let f be as in Theorem 1. Then for each n sufficiently large the polynomial $s_n(z) \equiv \sum_0^n a_k z^k$ is not the polynomial of degree n of best Tchebycheff approximation to f on Δ .*

Proof. Suppose that for some increasing sequence of integers m we have $s_m(z) \equiv p_m(z)$. Since

$$|a_{m+1}| + |a_{m+2}| - \sum_{m+3}^{\infty} |a_k| \leq \|f - s_m\|,$$

there follows from (1)

$$|a_{m+1}| + |a_{m+2}| \leq |a_{m+1}| [(1 + 4|a_{m+2}/a_{m+1}|^2)^{1/2} + A|a_{m+3}/a_{m+1}|]$$

and consequently

$$1 \leq 4|a_{m+2}/a_{m+1}| + A|a_{m+3}/a_{m+2}|,$$

where A is a constant independent of m . But the right-hand member of the last inequality approaches zero as $m \rightarrow \infty$, which gives the desired contradiction.

A result on the zeros of the error function will follow from

THEOREM 2. *If f is defined as in Theorem 1, then*

$$\lim_{n \rightarrow \infty} ((f(z) - p_n(z))/a_{n+1}z^{n+1}) = 1, \quad (3)$$

uniformly for z on each compact set in $|z| > 1$.

Proof. It is easy to show that

$$\lim_{n \rightarrow \infty} ((f(z) - s_n(z))/a_{n+1}z^{n+1}) = 1, \quad (4)$$

uniformly for z on each compact set in the plane. Since

$$\begin{aligned} \frac{1}{2\pi} \int_{|z|=1} |s_n(z) - p_n(z)|^2 dz &= \frac{1}{2\pi} \int_{|z|=1} |f(z) - p_n(z)|^2 dz - \sum_{n+1}^{\infty} |a_k|^2 \\ &\leq E_n(f)^2 - |a_{n+1}|^2 - |a_{n+2}|^2, \end{aligned}$$

it follows from (1) that

$$\frac{1}{2\pi} \int_{|z|=1} \left| \frac{s_n(z) - p_n(z)}{a_{n+1}z^{n+1}} \right|^2 |dz| \leq A_1 \left| \frac{a_{n+3}}{a_{n+1}} \right| + A_2 \left| \frac{a_{n+2}}{a_{n+1}} \right|^2 \rightarrow 0, \quad (5)$$

as $n \rightarrow \infty$. But $(s_n(z) - p_n(z))/a_{n+1}z^{n+1}$ is analytic for $|z| \geq 1$, even at ∞ , and so the last inequality implies that

$$\lim_{n \rightarrow \infty} \frac{s_n(z) - p_n(z)}{a_{n+1}z^{n+1}} = 0, \quad (6)$$

uniformly for z on each closed set in $|z| > 1$. From (4) and (6) we deduce (3).

By applying the Argument Principle to (3) one can establish

THEOREM 3. *Let f be as in Theorem 1 and let $\epsilon > 0$. Then for each $n > n_\epsilon$ the error function $f(z) - p_n(z)$ has exactly $n + 1$ zeros in $|z| < 1 + \epsilon$.*

Concerning the distribution of these zeros of the error function we prove

THEOREM 4. *Let f be as in Theorem 1 and for $n > n_1$ let $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_{n+1}^{(n)}$ be the $n + 1$ zeros of $f(z) - p_n(z)$ in the disk $|z| < 2$. Then*

$$\lim_{n \rightarrow \infty} \left| \prod_{i=1}^{n+1} (z - \alpha_i^{(n)}) \right|^{1/n} = |z|, \quad (7)$$

uniformly for z on each compact set in $|z| > 1$.

The limit (7) implies that for an arbitrary function F analytic on Δ the sequence of polynomials of respective degrees n found by interpolation to F in the points $\alpha_i^{(n)}$ converges maximally [6, Sect. 7.2] to F on Δ .

Proof. Write $f(z) - p_n(z) = \omega_n(z) g_n(z)$, where $\omega_n(z) \equiv \prod_{i=1}^{n+1} (z - \alpha_i^{(n)})$, and set $M_n \equiv \max_{|z| \leq 1} |\omega_n(z)|$. We shall show that

$$\limsup_{n \rightarrow \infty} M_n^{1/n} \leq 1, \quad (8)$$

which is equivalent to (7), see [6, Sect. 7.4].

Let $1 < \rho < R$. Since $|\omega_n(z)| \leq (R + 2)^{n+1}$ for $|z| = R$, it follows from Theorem 2 that

$$\frac{R}{R + 2} \leq \liminf_{n \rightarrow \infty} \min_{|z|=R} \left| \frac{g_n(z)}{a_{n+1}} \right|^{1/n}.$$

By the “minimum modulus principle” we have for n sufficiently large

$$\min_{|z|=R} |g_n(z)| \leq \min_{|z|=\rho} |g_n(z)|,$$

and so

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \max_{|z|=\rho} \left| \frac{\omega_n(z)}{z^{n+1}} \right|^{1/n} \liminf_{n \rightarrow \infty} \min_{|z|=R} \left| \frac{g_n(z)}{a_{n+1}} \right|^{1/n} \\ & \leq \limsup_{n \rightarrow \infty} \max_{|z|=\rho} \left| \frac{\omega_n(z) g_n(z)}{a_{n+1} z^{n+1}} \right|^{1/n} = 1. \end{aligned}$$

Hence

$$\limsup_{n \rightarrow \infty} M_n^{1/n} \leq \limsup_{n \rightarrow \infty} \max_{|z|=\rho} |\omega_n(z)|^{1/n} \leq \rho(R + 2)/R.$$

Letting $R \rightarrow \infty$ and $\rho \rightarrow 1$ in the last inequality we deduce (8). This proves Theorem 4.

We now show that under certain conditions the n -th error function has exactly $n + 1$ zeros in Δ .

THEOREM 5. *Let $f(z) = \sum_0^\infty a_k z^k$, where $n^{1/2} a_{n+1}/a_n \rightarrow 0$ as $n \rightarrow \infty$. Then for each n sufficiently large $f(z) - p_n(z)$ has exactly $n + 1$ zeros in $|z| < 1$ and no zeros on $|z| = 1$.*

Proof. It suffices to show that the limit (3) holds uniformly for $|z| = 1$.

By the Cauchy-Schwarz inequality and (5) we have for $|z| = 1$

$$\begin{aligned} \left| \frac{s_n(z) - p_n(z)}{a_{n+1} z^{n+1}} \right|^2 & \leq \frac{(n+1)}{2\pi} \int_{|z|=1} \left| \frac{s_n(z) - p_n(z)}{a_{n+1} z^{n+1}} \right|^2 |dz| \\ & \leq (n+1) \left[A_1 \left| \frac{a_{n+3}}{a_{n+1}} \right| + A_2 \left| \frac{a_{n+2}}{a_{n+1}} \right|^2 \right]. \end{aligned}$$

The hypothesis on the coefficients a_k implies that the right-hand member of the last inequality approaches zero as $n \rightarrow \infty$. Hence (6) and therefore (3) hold uniformly for $|z| = 1$. This proves Theorem 5.

Using the estimates obtained in this paper one can deduce from Rouché's theorem that for $n \geq 6$ the polynomial of degree n of best Tchebycheff approximation to e^z on Δ interpolates to e^z in exactly $n + 1$ points in Δ .

Added in proof. In contrast, S. Ja. Al'per has shown [Mathematical Analysis and its Applications (Russian), pp. 3-6. Izdat. Rostov. Univ., Rostov-na-Donu, 1969] that for each n there exists a function $f(z)$ (depending on n) which is analytic in $|z| < 1$ continuous on $|z| \leq 1$, and such that $f(z) - p_k(z) \neq 0$, z in Δ , $k \leq n-1$.

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